Introduction to fluid mechanics
Dimensional Analysis and Similitude

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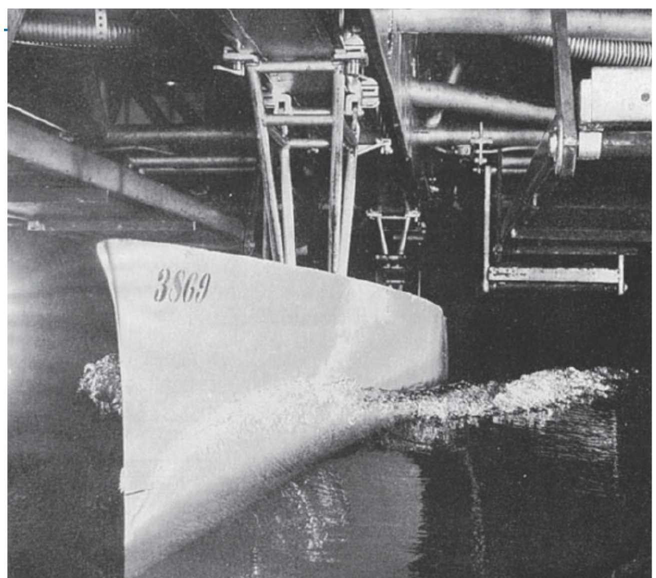
Need for Dimensional Analysis

- In fluid mechanics several variables are involved.
- Often they can be grouped in non-dimensional numbers.
- **“Universal” FUNCTIONAL RELATIONSHIP** between those non-dimensional numbers ↔ EXPERIMENTS.
- Applications:

  - **Design of experiments at reduced scales** (e.g. wind tunnel, water flumes)
  - **Simple models** (e.g., the ‘log-law’: logarithmic velocity profile)



Example: Drag Force on a Sphere

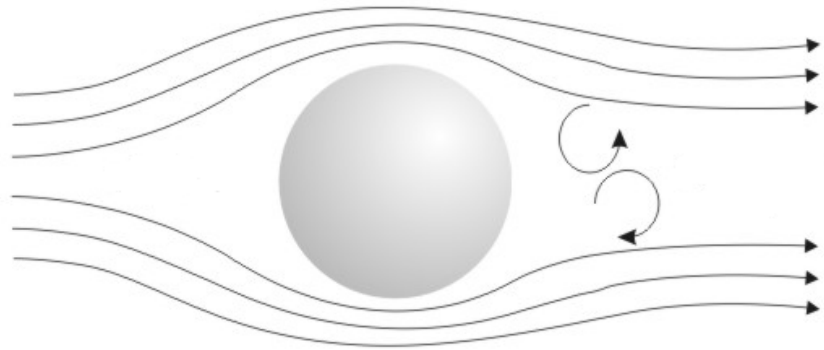
Important variables:

Viscosity: μ

Velocity: V

Diameter: D

Density: ρ



$$\underbrace{F_D}_{\text{dependent}} = f\left(\underbrace{V,\rho,\mu,D}_{\text{independent variables}}\right)$$

GOAL: Identify non-dimensional groups between the important variables and find a 'universal' relation between them

Introduction

Primary dimensions:

Table 1.2 PRIMARY DIMENSIONS		
Dimension	**Symbol**	**Unit (SI)**
Length	L	meter (m)
Mass	M	kilogram (kg)
Time	T	second (s)
Temperature	θ	kelvin (K)
Electric current	i	ampere (A)
Amount of light	C	candela (cd)
Amount of matter	N	mole (mol)

Secondary dimensions:

Force:

$$[F]=[ma]=M\frac{L}{T^2}=\frac{ML}{T^2}$$

Pressure:

$$[p]=\left[\frac{F}{A}\right]=\frac{ML/T^2}{L^2}=\frac{M}{LT^2}$$

Viscosity:

$$\tau=\mu\frac{du}{dy}$$

$$\frac{ML}{T^2}\cdot\frac{1}{L^2}=[\mu]\frac{L}{TL}\rightarrow[\mu]=\frac{M}{TL}$$

Buckingham Π Theorem

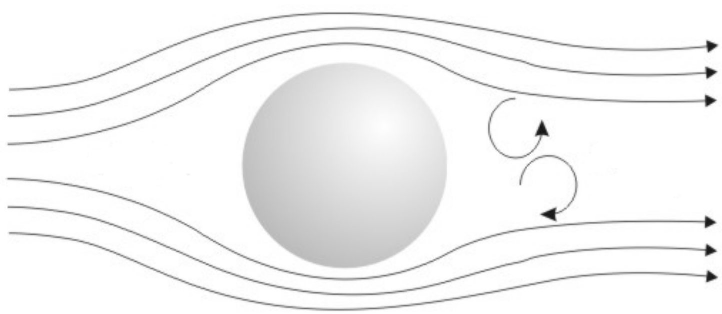
- *n:* number of variables involved
- *m:* number of basic dimensions included in the variables

- (n - m) **number of dimensionless variables (Π groups)**

Example: Drag Force on a Sphere

- *n:* number of variables involved: $\left(F_D, V, \rho, \mu, D\right)$
- *m:* number of basic dimensions included in the variables: $\left(\left[M, L, T\right]\right)$
- (*n* - *m*) **number of dimensionless variables (Π groups)**

- In this case: 5-3=2, number of dimensionless variables



Example: Drag Force on a Sphere

Important variables:

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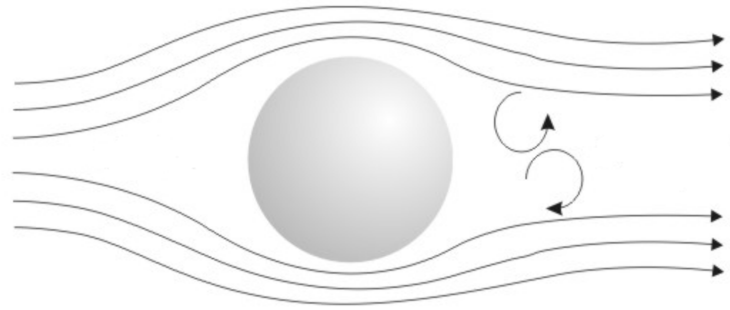


$$\underbrace{F_D}_{\text{dependent}} = f\left(\underbrace{V, \rho, \mu, D}_{\text{independent variables}}\right)$$

$$[F_D] = \frac{ML}{T^2}; \quad [V] = \frac{L}{T}; \quad [\rho] = \frac{M}{L^3}; \quad [\mu] = \frac{M}{LT}; \quad [D] = L$$

Drag Force on a Sphere

- *Exponent Method*

1. Dimensions of significant variables are

$$[F]=\frac{ML}{T^2},\ [V]=\frac{L}{T},\ [\rho]=\frac{M}{L^3},\ [\mu]=\frac{M}{LT},\ [D]=L$$

2. Number of π-groups is $5-3=2$.

3. Form product with dimensions.

$$\frac{ML}{T^2}=\left[\frac{L}{T}\right]^a\times\left[\frac{M}{L^3}\right]^b\times\left[\frac{M}{LT}\right]^c\times[L]^d$$

$$=\frac{L^{a-3b-c+d}M^{b+c}}{T^{a+c}}$$

4. Dimensional homogeneity. Equate powers of dimensions on each side.

$$L:\quad a-3b-c+d=1$$

$$M:\quad b+c=1$$

$$T:\quad a+c=2$$

5. Solve for exponents *a, b,* and *c* in terms of *d*.

$$\begin{pmatrix}1 & -3 & -1\\ 0 & 1 & 1\\ 1 & 0 & 1\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}=\begin{pmatrix}1-d\\ 1\\ 2\end{pmatrix}$$

The value of the determinant is -1 so a unique solution is achievable. Solution is $a=d,\quad b=d-1,\quad c=2-d$

6. Write dimensional equation with exponents.

$$F=V^d\rho^{d-1}\mu^{2-d}D^d$$

$$F=\frac{\mu^2}{\rho}\left(\frac{\rho VD}{\mu}\right)^d$$

$$\frac{F\rho}{\mu^2}=\left(\frac{\rho VD}{\mu}\right)^d$$

There are two π-groups:

$$\pi_1=\frac{F\rho}{\mu^2}\ \text{and}\ \pi_2=\frac{\rho VD}{\mu}$$

By dividing π_1 by the square of π_2, the π_1 group can be written as $F_D/(\rho V^2D^2)$, so the functional form of the equation can be written as

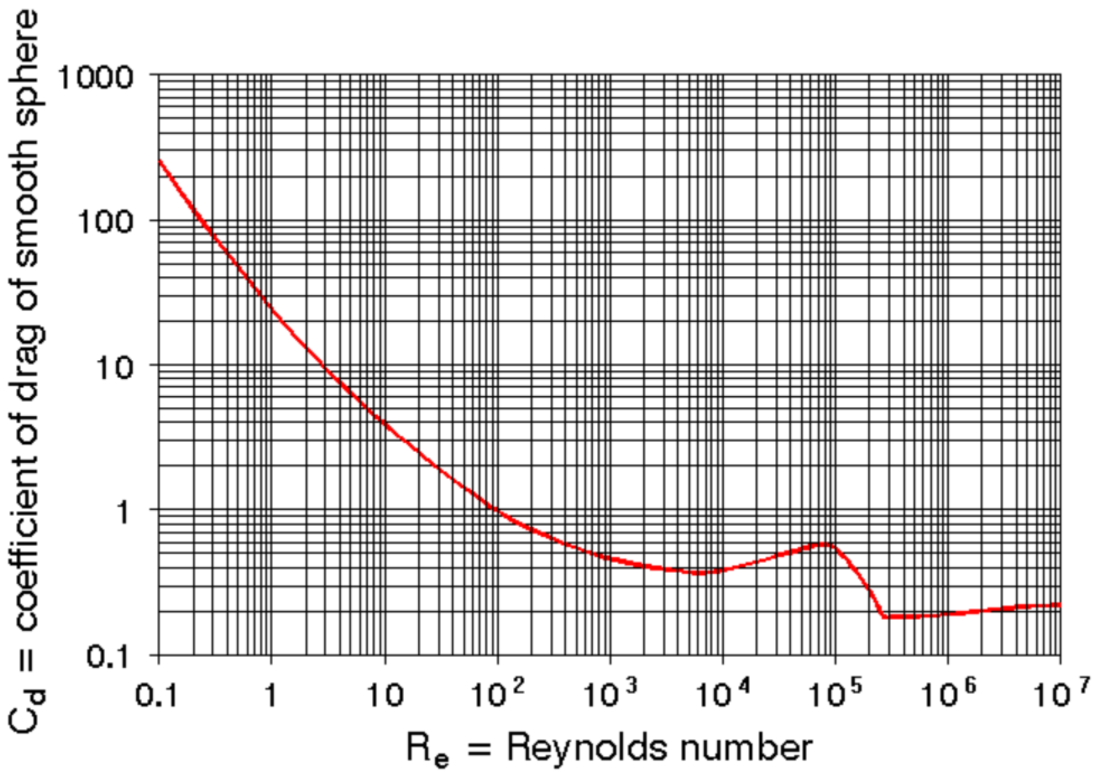
$$\frac{F}{\rho V^2D^2}=f\left(\frac{\rho VD}{\mu}\right)$$

Drag Force on a Sphere

Empirical 'universal' relation between the two non-dimensional groups (from experiments):

$$C_d = \frac{F_D}{\frac{1}{2}\rho V^2 \left(\frac{\pi}{4} D^2\right)}$$

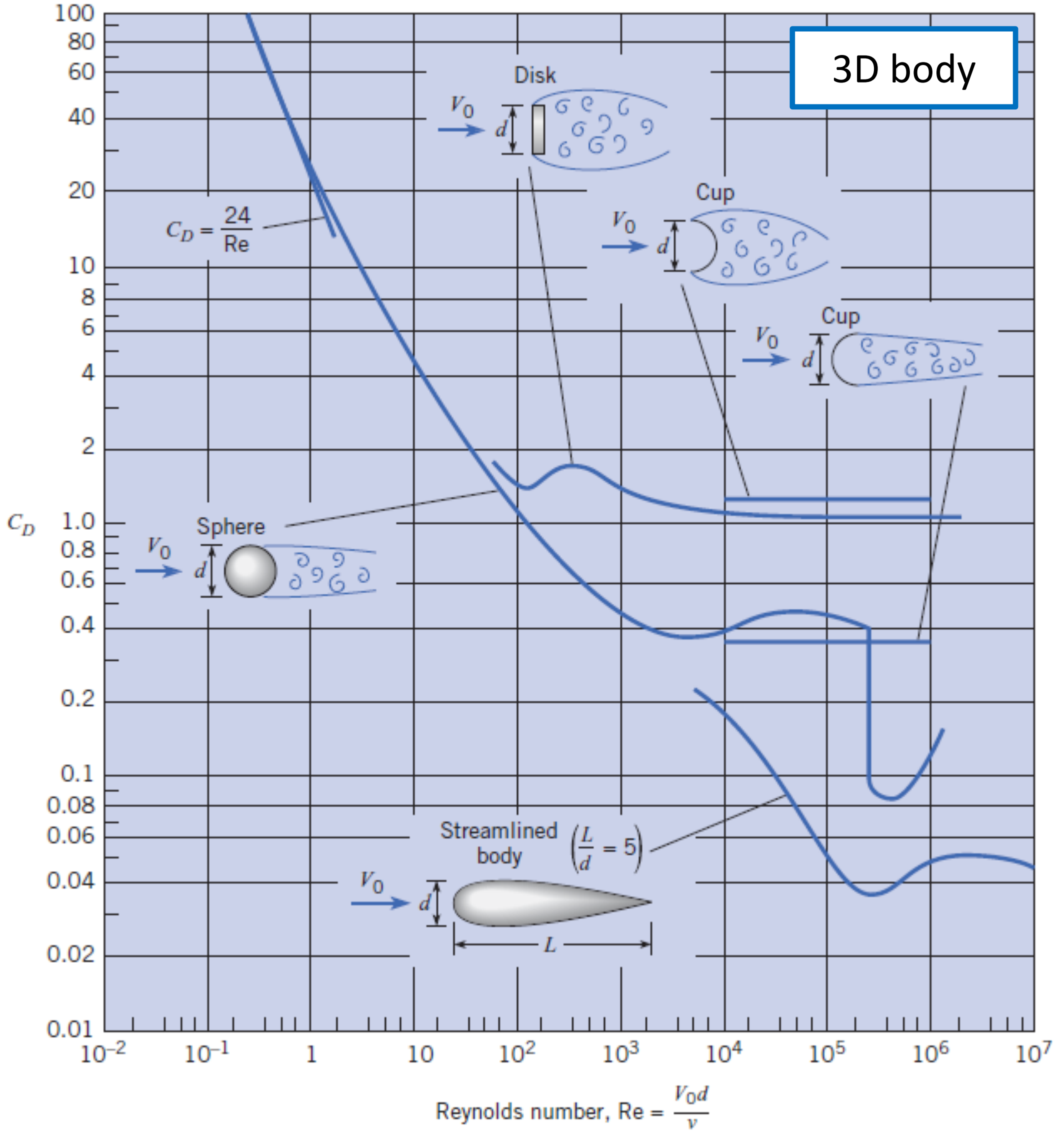
(drag coefficient)



$$\frac{\rho D V}{\mu} = \mathrm{Re} \;\; \text{(Reynolds number)}$$

Drag coefficient

(We will see in detail later on - Chapter 11)

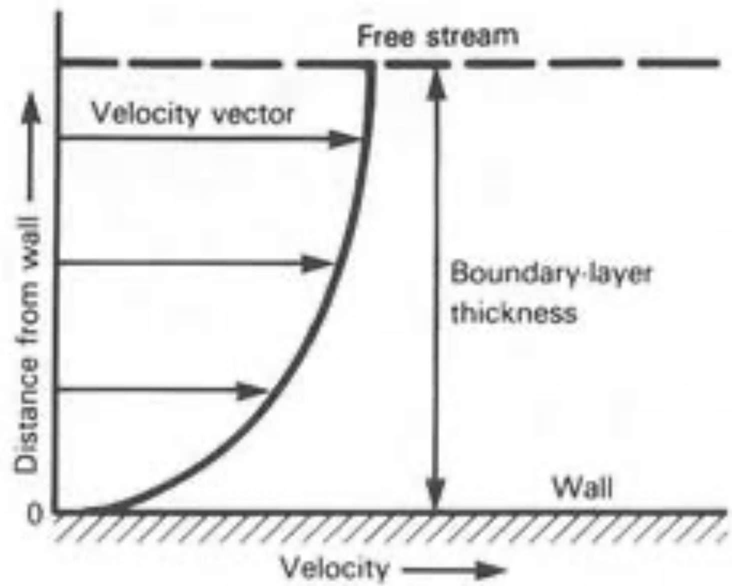


$$F_D = C_D A\left(\frac{\rho V_0^2}{2}\right)$$

EXERCISE: log-law in turbulent boundary layer flow

In the surface layer (lowest 10-20%), the **velocity gradient** is known to change **with height, density and shear stress**

$$\frac{d\bar{u}}{dz} = f(\rho, z, \tau)$$



Using dimensional analysis, show that the velocity profile should be LOGARITHMIC.

$$\left[\frac{d\bar{u}}{dz}\right] = \frac{1}{T}; [\rho] = \frac{M}{L^3}; [z] = L;$$

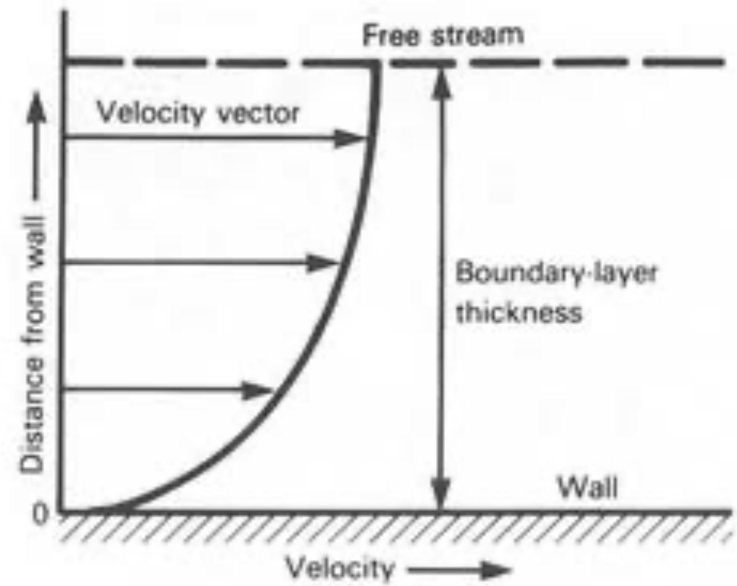
$$[\tau] = \frac{ML}{T^2L^2} = \frac{M}{T^2L}$$

$$n = 4, m = \underbrace{3}_{(M,L,T)} \rightarrow n - m = 1$$

1 dimensionless group!

Turbulent boundary layer flow:

The velocity gradient changes,



$$\frac{1}{T} = \left[\frac{M}{L^3}\right]^a \times [L]^b \times \left[\frac{M}{T^2 L}\right]^c$$

L: $0 = -3a + b + c$
M: $0 = a + c$
T: $-1 = -2c$

$c = 1/2; \quad a = -1/2; \quad b = -1$

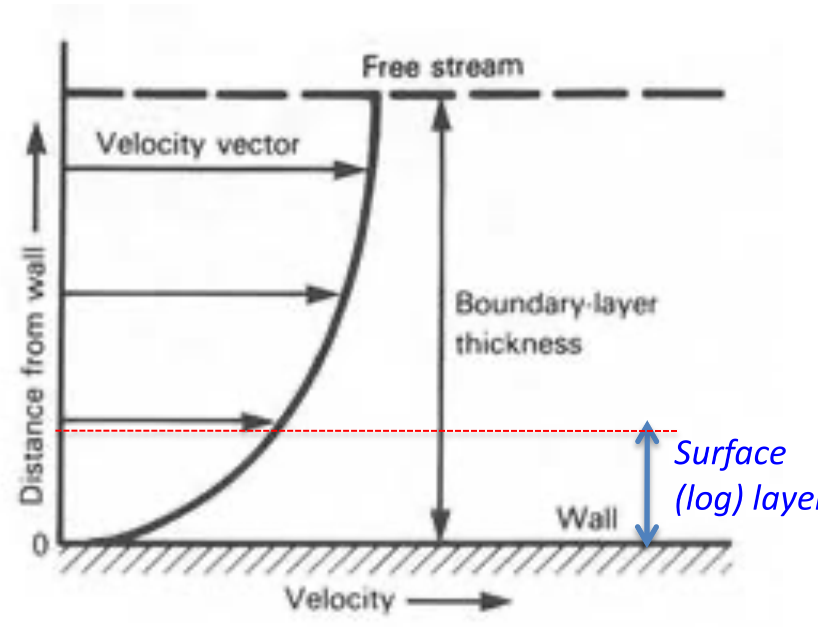
$$\frac{d\bar{u}}{dz} \propto \rho^{-1/2} z^{-1} \tau^{1/2} = \frac{\sqrt{\tau/\rho}}{z}$$

$$\left[\sqrt{\tau/\rho}\right] = \frac{L}{T} \rightarrow \textit{friction velocity}: u_* = \sqrt{\tau/\rho}$$

Turbulent boundary layer flow

$$\frac{z}{u_*}\frac{d\bar{u}}{dz} = \text{constant (universal)}$$

$$\frac{z}{u_*}\frac{d\bar{u}}{dz} = \frac{1}{\kappa} = \frac{1}{0.4}, \quad \kappa : von\ Karman\ cont.$$



$$\int_{u_1}^{u_2} d\bar{u} = \frac{u_*}{\kappa}\int_{z_1}^{z_2}\frac{dz}{z}$$

$$u_2 - u_1 = \frac{u_*}{\kappa}\ln\frac{z_2}{z_1} \rightarrow \quad at\ z_1 = z_0, \bar{u}_1 = 0 \rightarrow \boxed{\bar{u}(z) = \frac{u_*}{\kappa}\ln\frac{z}{z_0}}$$

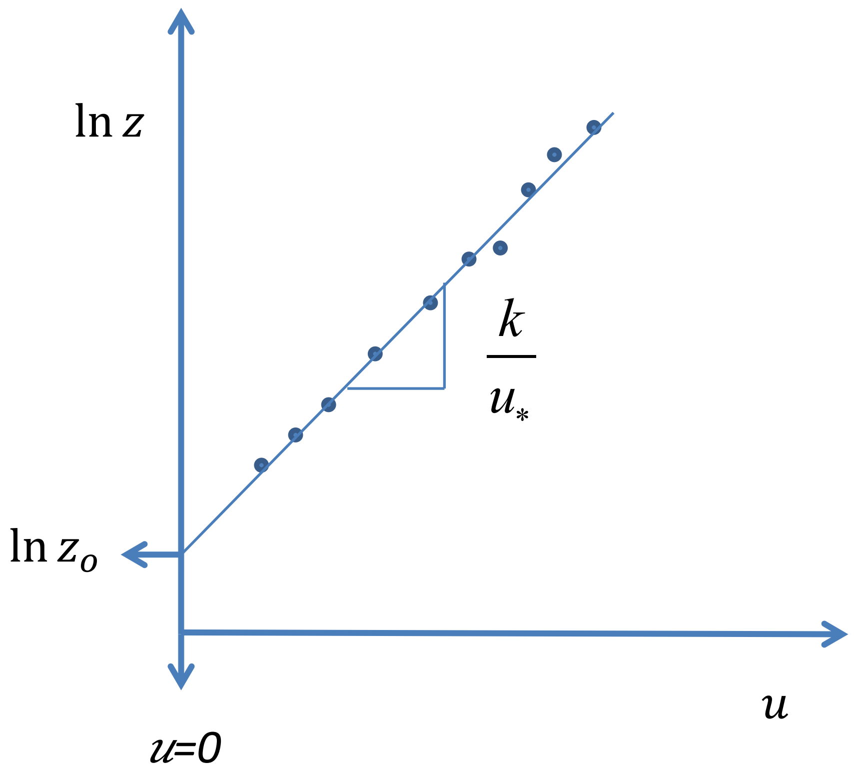
Over a rough surface:
z_o is the aerodynamic roughness

Logarithmic velocity profile (log-law) *in the surface layer*

z_o (aerodynamic/hydrodynamic roughness): defined as the height at which the velocity would be zero if extrapolating the log law.

The LOG LAW is valid in the 'surface layer' (the lowest 10-20% of the boundary layer), where u_* can be assumed to be constant

Turbulent boundary layer flow:



Significant variables and common dimensionless numbers:

$$\Delta p = f\left(V, L, \rho, \mu, E_v, \sigma, \Delta\gamma\right)$$

$$\left[\Delta p\right] = \frac{F}{L^2} \quad \textit{pressure difference}$$

$$\left[\Delta\gamma\right] = \frac{F}{L^3} \quad \textit{weigth difference}$$

$$\left[V\right] = \frac{L}{T} \quad \textit{velocity}$$

$$\left[\rho\right] = \frac{FT^2}{L^4} = \frac{M}{L^3} \quad \textit{density}$$

$$\left[L\right] = L \quad \textit{length}$$

$$\left[\mu\right] = \frac{FT}{L^2} \quad \textit{viscosity}$$

$$\left[\sigma\right] = \frac{F}{L} \; \textit{surface tension}$$

$$\left[E_v\right] = \frac{F}{L^2} \quad \textit{elasticity}$$

$$\frac{\Delta p}{\rho V^2} = g\left(\frac{VL\rho}{\mu}, \frac{V}{\sqrt{\frac{E_v}{\rho}}}, \frac{\rho L V^2}{\sigma}, \frac{V^2}{L\frac{\Delta\gamma}{\rho}}\right)$$

Frequent dimensionless numbers:

Table 8.3 COMMON Π-GROUPS

π-Group	Symbol	Name	Ratio
$\frac{p - p_0}{(\rho V^2)/2}$	C_p	Pressure coefficient	$\frac{\text{Pressure difference}}{\text{Kinetic pressure}}$
$\frac{\tau}{(\rho V^2)/2}$	c_f	Shear-stress coefficient	$\frac{\text{Shear stress}}{\text{Kinetic pressure}}$
$\frac{F}{(\rho V^2 L^2)/2}$	C_F	Force coefficient	$\frac{\text{Force}}{\text{Kinetic force}}$
$\frac{\rho L V}{\mu}$	Re	Reynolds number	$\frac{\text{Kinetic force}}{\text{Viscous force}}$
$\frac{V}{c}$	M	Mach number	$\frac{\text{Kinetic force}}{\text{Compressive force}}$
$\frac{\rho L V^2}{\sigma}$	We	Weber number	$\frac{\text{Kinetic force}}{\text{Surface-tension force}}$
$\frac{V}{\sqrt{gL}}$	Fr	Froude number	$\frac{\text{Kinetic force}}{\text{Gravitational force}}$

Interpretation: Force ratios

Inertial (kinetic) force: $\left[F_k\right]=\left[\frac{1}{2}\rho V^2 \cdot A\right] \propto \left(\rho V^2\right)\left(L^2\right)=\rho L^2 V^2$

Viscous force: $\left[F_v\right]=\left[\tau A\right] \propto \left(\mu \frac{V}{L}\right)\left(L^2\right)=\mu V L$

$$\frac{F_k}{F_v} \propto \frac{\rho L^2 V^2}{\mu V L} = \frac{\rho L V}{\mu} = \text{Re} \quad \text{(Reynolds number)}$$

<u>Laminar flows</u>

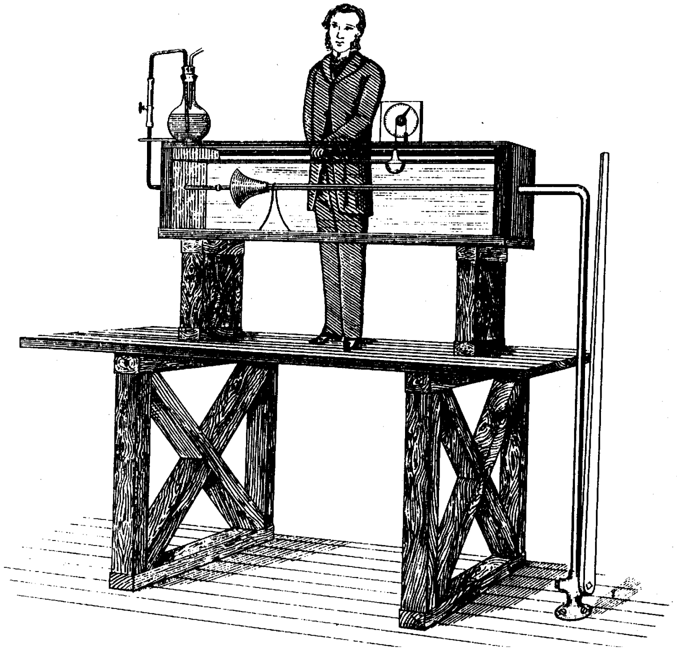
Example:
- water flow inside soils

<u>Turbulent flows</u>

Examples:
- river flows
- atmospheric boundary layer
- most pipe and channel flows

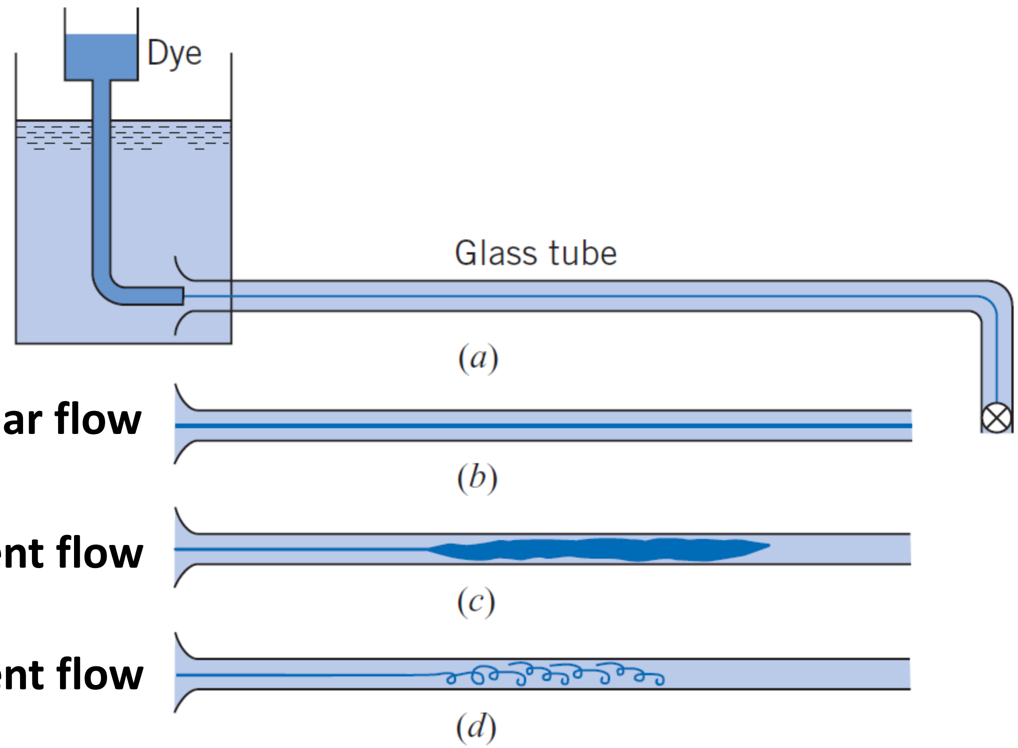
Laminar and Turbulent Flow

Reynolds' experiment



Reynolds Number: $\mathrm{Re} = \dfrac{VD}{\nu} = \dfrac{\rho VD}{\mu}$

$\mathrm{Re} \leq 2000$	laminar flow
$2000 \leq \mathrm{Re} \leq 3000$	unpredictable
$\mathrm{Re} \geq 3000$	turbulent flow



Interpretation: Force ratios

Kinetic (inertial) force: $\left[F_k\right]=\left[\frac{1}{2}\rho V^2 \cdot A\right]\propto\left(\rho V^2\right)\left(L^2\right)=\rho L^2 V^2$

Pressure force: $\left[F_p\right]=\left[\Delta p\, A\right]\propto \Delta p\, L^2$

$$\frac{2F_p}{F_k}\propto\frac{2\Delta p\, L^2}{\rho L^2 V^2}=\boxed{\frac{\Delta p}{\frac{1}{2}\rho V^2}=C_p}\quad \text{(pressure coefficient)}$$

Interpretation: Force ratios

Kinetic force: $$[F_k] = \left[\frac{1}{2}\rho V^2 \cdot A\right] \propto (\rho V^2)(L^2) = \rho L^2 V^2$$

Surface tension force : $$[F_\sigma] \propto \sigma L$$

$$\frac{F_k}{F_\sigma} \propto \frac{\rho L^2 V^2}{\sigma L} = \boxed{\frac{\rho L V^2}{\sigma} = \mathrm{We}} \ \text{(Weber number)}$$

$We >> 1$ → Surface tension not important

Interpretation: Force ratios

Inertial (kinetic) force: $\left[F_k\right]=\left[\frac{1}{2}\rho V^2\cdot A\right]\propto\left(\rho V^2\right)\left(L^2\right)=\rho L^2V^2$

Gravity force : $\left[F_g\right]=\left[mg\right]\propto\left(\rho L^3\right)\left(g\right)$

$$\sqrt{\frac{F_k}{F_g}}\propto\sqrt{\frac{\rho L^2V^2}{\rho L^3 g}}=\sqrt{\frac{V^2}{Lg}}=\boxed{\frac{V}{\sqrt{Lg}}=\mathrm{Fr}}\qquad\text{(Froude number)}$$

- Note: Very important in **Hydraulics (open channel flow)** – will study in Chapter 15

Interpretation: Force ratios

Inertial (kinetic) force: $$\left[F_k\right] = \left[\frac{1}{2}\rho V^2 \cdot A\right] \propto \left(\rho V^2\right)\left(L^2\right) = \rho L^2 V^2$$

Elastic force: $$\left[F_c\right] = \left[E_v L^2\right] = \rho V c L^2$$

$$\frac{F_k}{F_c} \propto \frac{\rho L^2 V^2}{\rho V c L^2} = \boxed{\frac{V}{c} = \mathrm{M}}\ (\text{Mach number})$$

Subsonic flow (incompressible)

$$M < 0.8$$

Most environmental flows:

-rivers
-lakes
-groundwater
-atmosphere

Supersonic flow (compressible!)

$$M > 1.2$$

Experimental Fluid Mechanics:

Experiments are often difficult in ‘real’ (field) scales because of:

- Complicated boundary conditions
- Difficult to measure the flow in the field
- Turbulence
- Non-repeatable

An alternative is to use **laboratory experiments** (e.g., in wind tunnels or water flumes).

- **Two aspects that we have to take into account in lab experiments:**
  - Geometric similarity
  - Dynamic similarity

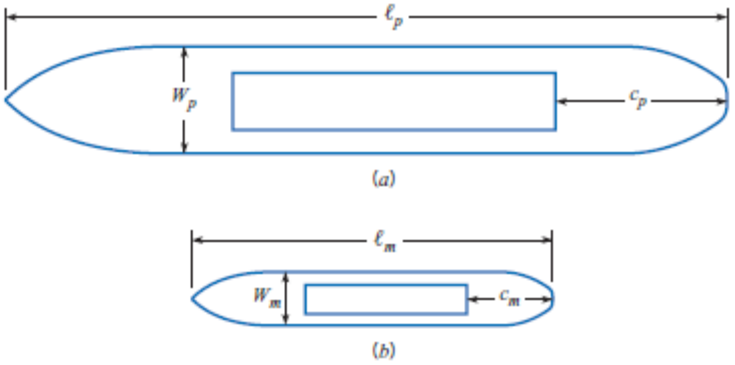
Experimental Fluid Mechanics:

1. Geometric similarity

means that the model is an exact geometric replica of the prototype

Figure 8.4

(a) Prototype. (b) Model.



$$\frac{L_m}{L_p} = \frac{W_m}{W_p} = \frac{c_m}{c_p}$$

2. Dynamic similarity

the forces that act on corresponding masses in the model and prototype are in the same ratio throughout the entire flow field.

$$\frac{F_{i,p}}{F_{g,p}} = \frac{F_{i,m}}{F_{g,m}} \rightarrow Fr_p = Fr_m$$

$$\frac{F_{i,p}}{F_{v,p}} = \frac{F_{i,m}}{F_{v,m}} \rightarrow \mathrm{Re}_p = \mathrm{Re}_m$$